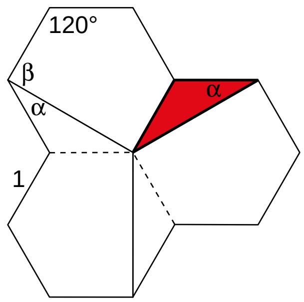
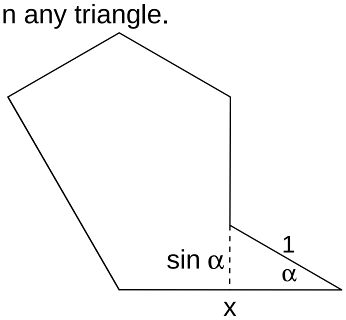
Worksheet 1

Answers to all of the worksheet problems are available at http://www.tessellations.com.

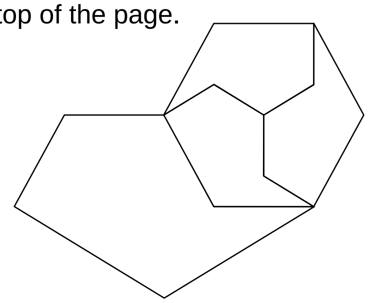
This worksheet explores the angles, edge lengths, and scaling of the tiles in the puzzle. The dashed lines below show how the tile is related to a hexagon. How does the area of the tile compare to the area of the hexagon?



If the tile's short edges and the hexagon's edges have length 1, how does the area of the tile compare to the area of an equilateral triangle with edge length 1? Given that the interior angles of a hexagon are each 120°, what are the values of the angles α and β? Hint: remember the sum of the three angles in any triangle.



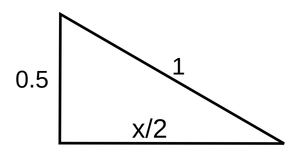
Given that the sine of α is 0.5, use the Pythagorean theorem to deduce the value of x, the scaling factor between the short and long edges. Note that x is also the linear scaling factor between two successive generations of tiles. The area between two successive generations of tiles will scale as x^2. Using this fact, determine x by comparing the figure below to the one at the top of the page.



A: The area of a HexaPlex tile is the same as that of a regular hexagon with edge length equal to the short edges of a tile.

A: A regular hexagon with edges of length 1 can be divided into six equilateral triangles with edges of length 1, so the tringle has 1/6 the area of a hexagon or a tile.

In the red triangle, the large angle is 120° and the two small angles are the same. All three sum to 180°, so each small angle α must be 30°. The angle β is given by 120° - α = 90°



A: From the Pythagorean theorem, $(x/2)^2 + 0.5^2 = 1^2$, or $x^2/4 + 1/4 = 1$, from which $x^2 = 3$ and $x = \sqrt{3}$.

Since the three smaller tiles form a hexagon with area equal to that of the larger tile, each larger tile must have 3 times the area of a smaller tile. The linear scaling factor x is the square root of the area scaling factor, so $x = \sqrt{3}$.

Worksheet 2

This worksheet explores series of numbers. Look at the first fractal tessellation in the instructions, the one with a wave-like boundary. Write down the number of 1st generation tiles, 2nd generation tiles, ... through 5th generation tiles in upper left corners of the cells in the table below. Do the same for the second fractal tessellation, the one with a broccoli-like boundary. What can you say about the trend in the number of tiles with each successive generation?

A: The counts are shown in the table. The number of tiles increases with each successive generation.

number / area	1st Gen	2nd Gen	3rd Gen	4th Gen	5th Gen
Wave tess.	3 / 3	9 / 3	27 / 3	63 / 2.33	135 / 1.67
Broccoli tess.	3 / 3	12 / 4	30 / 3.33	66 / 2.44	138 / 1.70
Tripling	3 / 3	9 / 3	27 / 3	81 / 3	273 / 3

From Worksheet 1, we know that the area of a tile decreases by a factor of three from one generation to the next smaller generation. Using this fact, calculate the area occupied by all of the second generation tiles in these two fractal tessellations and compare it to the area occupied by the first generation tiles. For simplicity, call the area of a single first generation tile 1. (The area occupied by all of the tiles in a given generation is just the area occupied by one tile times the number of tiles.) Fill in the lower right corner of each cell with the area occupied by each generation in each of the two tessellations. Which of the two tessellations do you think has the larger total area (when taken through an infinite number of iterations)?

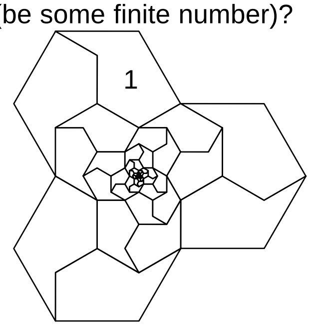
A: The areas are shown in the table. Since the area is larger for each generation, it appears that the broccoli tessellation will have the greater total area. It's not clear that the area for each generation will continue to be larger for the broccoli tessellation as the tiles get smaller, but the difference in areas through the first few generations is so large that it's a safe bet that the wave tessellation area will never catch up.

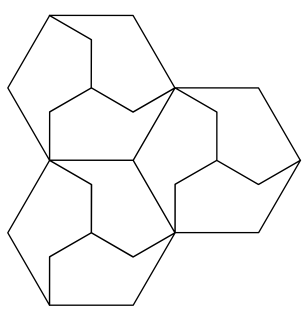
If the number of tiles tripled with each successive generation, as shown in the last row of the table, the area occupied by each generation would be the same. What would this imply for the area of the infinite tiling? If you summed the areas for all generations through infinity for both the wave-like tessellation and the broccoli-like tessellation, would the sum diverge (go to infinity) or converge (be some finite number)?

A: The area for the infinite tessellation in this case would be an infinite sum of 3's, so the area would be infinite.

For the broccoli and wave tessellations, it is clear from a visual inspection that the infinite tessellation will be finite in area and in fact, as drawn, would not extend beyond the boundaries of the page. The area sums would thus converge to finite numbers.

For the collection of tiles shown at right, given the area of one of the large tiles is 1, what is the total area of the infinite number of tiles shown?





A: The same area can be covered by nine large tiles, so the total area of this infinite collection of tiles is 9. Writing this as an infinite series:

$$6 + \frac{6}{3^1} + \frac{6}{3^2} + \frac{6}{3^3} + \ldots = 9$$

Worksheet 3

This worksheet explores 5-hexes - shapes made up by connecting five hexagons in edge-to-edge fashion. There are 22 distinct 5-hexes (not counting mirrored variants). Using the grid below, see how many of these you can find. To help you get started, one is shown. Write the letter "m" within each 5-hex that is not changed by mirroring; i.e., that possesses mirror symmetry about some line. Write the letter "r" within each 5-hex that possesses rotational symmetry about its center. You may want to copy this page before starting so that you have more space to work.

A: The 22 5-hexes ares shown below, with the markings for mirror and reflection symmetry.